Product Numbers

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The 'balancing problem' asks us to find a pair of natural numbers m, n (with $m < n$) such that the sum of all the natural numbers from 1 till $m-1$ equals the sum of all the natural numbers from $m+1$ till n. (We can think of this as a kind of balancing problem. The number m acts as a point of balance.) This problem was first published in the English magazine *Strand* in December 1914. It was solved by P. C. Mahalanobis by trial and error; Ramanujan came up with a general solution. The simplest solution is

$$1 + 2 + 3 + 4 + 5 = 7 + 8.$$

Here the middle term (the point of balance) is $m = 6$, and $n = 8$. The problem has infinitely many solutions.

Product balance problem

In just the same way, we formulate a *product balance problem*. That is, we seek a pair of natural numbers m, n (with $m < n$) such that the product of all the natural numbers from 1 till $m - 1$ equals the product of all the natural numbers from $m + 1$ till n.

For example, we may take $m = 7$ and $n = 10$. It may be checked that

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 8 \times 9 \times 10 \quad \text{(both sides are equal to 720).}$$

But trying to find more such pairs of numbers gets difficult - and this is because there are no other solutions! Indeed, (7, 10) is the only such pair.

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Proof of claim

We may check by looking at all the cases that there exists no solution with $m < 7$. And we already know that $m = 7$, $n = 10$ is a solution.

Now suppose there exists a pair (m, n) satisfying the condition of the problem, with $m > 7$. Let p be the largest prime number less than m, so $p \geq 7$. Then the product of the natural numbers from 1 till $m - 1$ is necessarily a multiple of p.

Hence the product of the natural numbers from $m + 1$ till n must also be a multiple of p. For this to happen, there must be a multiple of p among the numbers from $m + 1$ to n. Therefore $n \geq 2p$, as the least natural number after p which is a multiple of p is $2p$.

Now it is known that for any natural number $k \geq 2$, there exists a prime number q such that $k < q < 2k$. For more on this remarkable result, which is known in number theory as *Bertrand's postulate,* please see this reference: https://en.wikipedia.org/wiki/Bertrand's_postulate.

Stronger results are known. Ramanujan showed that for any natural number $k \geq 6$, there exist *at least two prime numbers* q_1, q_2 such that $k < q_1 < q_2 < 2k$.

Applying Ramanujan's result to the present situation, we see that there exist two prime numbers q_1, q_2 such that $p < q_1 < q_2 < 2p$. Of these two primes, it might happen that $q_1 = m$. (This will be the case if m itself is a prime number. If not, we will have $q_1 > m$.) But in any case we will have $q_2 > m$. This means that the product of the natural numbers from $m + 1$ till n must be divisible by q_2.

But the product of the natural numbers from 1 to $m - 1$ cannot be divisible by q_2, as q_2 is a prime number and $q_2 \geq m$. Hence it cannot be that the product of all the natural numbers from 1 till $m - 1$ equals the product of all the natural numbers from $m + 1$ till n.

Therefore there exists no solution other than $m = 7$, $n = 10$.

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Bihar Teacher's Innovative Technique of Teaching Math Tables

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Other readers have contributed many similar tricks in the comment section. As math-o-philes, our readers would be part of the group which immediately analyses why this trick works. We invite our readers to share their thoughts on how math tricks should be taught in class, given their unmistakable magnetism. Write in to AtRiA.editor@apu.edu.in